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COMMUTATION RELATIONS AND RELATED TOPICS

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ABSTRACT. We shall describe some aspects of commutation relations.

1. Introduction.

H.Umegaki[12] established the foundation of the noncommutative probability theory. The concept of covariance has not been focused in this setting. Recently M.Fujii,T.Furuta, R.Nakamoto and S.Takahashi[3] extensively studied noncommutative covariance with some applications for several inequalities. Related results are also studied in [4,11].

J.I.Fujii introduced the covariances and the variances for operators as follows. Let H be a separable Hilbert space and $x \in H, \|x\| = 1$. The covariance of (not necessarily bounded) operators A and B in a state x is defined by

$$Cov_x(A, B) = (A^*Bx, x) - (A^*x, x)(Bx, x),$$

and the variance of T in a state x is defined by

$$Var_x(A) = \|Ax\|^2 - |(Ax, x)|^2.$$

The following covariance -variance inequality is established in [3]. The inequality is a fundamental result in noncommutative probability.

Theorem 1.1(The covariance-variance inequality). *The square of the absolute of the covariance of operators A and B is not greater than the product of the variance of A and B:*

$$|Cov_x(A, B)|^2 \leq Var_x(A) \cdot Var_x(B).$$

We shall point out that the covariance variance inequality is exactly the generalized Schrödinger inequality. We briefly sketch the history of the uncertainty relation [1,5]. The Heisenberg relation

$$\Delta x \cdot \Delta p \sim \hbar$$

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is expressing the uncertainty principle in quantum physics. This relation was discussed by Heisenberg [6] in 1927. The derivation of the uncertainty principle from the mathematical formalism of wave mechanics was given by Kennard [7] as follows:

$$\frac{\hbar}{2} \leq \Delta x \cdot \Delta p.$$

Afterward Robertson [9] proved the general uncertainty principle, valid for an arbitrary pair of the selfadjoint operators A and B. The Robertson inequality reads:

$$\frac{\hbar}{2}| < C > | \leq \Delta A \cdot \Delta B,$$

where the operator C is defined as

$$C = (AB - BA)/i\hbar,$$

the brackets $<>$ denote the average values and the standard deviations are defined by the formula

$$(\Delta A)^2 = < (A - < A >)^2 > .$$

An improved inversion of the Robertson inequality is due to Schrödinger [10]. Schrödinger generalized Robertson's derivation by introducing $D = A + \alpha B + i\beta B$, where α and β are real numbers, and deducing from $0 \leq (Dx, Dx)$ the inequality expressing the nonpositivity of the discriminant ; by an appropriate choice of α and β he then obtained the following Schrödinger inequality:

$$(\frac{1}{2} < AB + BA > - < A >< B >)^2 + \frac{\hbar^2}{4}| < C > |^2 \leq (\Delta A)^2 \cdot (\Delta B)^2.$$

In this paper we shall point out that the generalized Schrödinger inequality is exactly the covariance variance inequality. We also look at the Hilbert C^*-module case. Finally we also look at the q -Fock space case and describe its relation to q-Kantrovich inequality.

2. Schrödinger inequality.

In this section we shall point out that the absolute value of the covariance of two selfadjoint operators is represented by the term of the Schrödinger inequality. .

Theorem 2.1. *Let A and B be (not necessarily bounded) selfadjoint operators on a Hilbert space H. Let $D(AB)$ and $D(BA)$ be the domain of AB and BA respectively. Let $x \in D(AB) \cap D(BA)$ and $\|x\| = 1$. Let $\{A, B\}$ be the Jordan product $AB + BA$. Let $[A, B]$ be the commutator $AB - BA$. Then*

$$|Cov_x(A,B)|^2 = \{\frac{1}{2}(\{A,B\}x,x) - (Ax,x)(Bx,x)\}^2 + \{\frac{1}{2}([A,B]x,x)\}^2.$$

Proof.

$$ReCov_x(A,B) = \frac{1}{2}(\{A,B\}x,x) - (Ax,x)(Bx,x),$$

$$ImCov_x(A,B) = \frac{1}{2i}([A,B]x,x).$$

Therefore using by

$$|Cov_x(A,B)|^2 = |ReCov_x(A,B)|^2 + |ImCov_x(A,B)|^2,$$

this theorem holds.

3 The case of Hilbert C^*-module.

Let X be a Hilbert C^*-modules over a C^*-algebra A.

$$\mathcal{L}_A(X_A) = \{T|T : X \to X, \exists T^*, T(x \cdot a) = (Tx)a\}.$$

The inner product $(x|y)$ has the following properties ;

$$(x|y \cdot a) = (x|y)a \quad \text{for} \quad x, y \in X, a \in A,$$

$$(x \cdot a|y) = a^*(x|y).$$

Then we shall define the covariance $Cov_x(S,T)$ for $S,T \in \mathcal{L}_A(X_A), x \in X$ as follows ;.

$$Cov_x(S,T) = (Sx - x(x|Sx)_A|Tx - x(x|Tx)_A)$$

$$Var_x(S) = Cov_x(S,S).$$

Then

$$Cov_x(S,T) = (Sx|Tx) - (Sx|x(x|Tx)_A) - (X(x|Sx)_A|Tx) + (x(x|Sx)_A|x(x|Tx)_A)$$

$$= (T^*Sx|x) - (Sx|x)(x|Tx)_A - x|Sx)^*(x|x)(x|Tx)_A$$

(We assume the following condition here.

$$(x|x) = 1.)$$

$= (T^*Sx|x) - (Sx|x)(x|Tx) - (Sx|x)(x|Tx) + (Sx|x)(x|Tx)$
$= (T^*Sx|x) - (Sx|x)(x|Tx).$
Then we have

$$Cov_x(S,T) = (T^*Sx|x) - (Sx|x)(x|Tx).$$

On the other hand, $Cov_x(S,T)$ is linear for S and conjugate linear for T,where $S,T \in \mathcal{L}_A(X_A)$. Thus $Cov_x(S,T)$ is an inner product for $\mathcal{L}_A(X_A)$. Therefore

$$\|Cov_x(S,T)\|^2 \leq \|Var_x(S)\| \cdot \|Var_x(T)\|.$$

Remark 3.1. *The condition $(x|x) = 1$ is different from the condition $\|x\| = 1$.*
The real part and the imaginary part of $Cov_x(S,T)$ are as follows;

$$\Re Cov_x(S,T) = \frac{1}{2}((S^*T + T^*S)x|x) - \Re((Tx|x)(x|Sx)),$$

$$\Im Cov_x(S,T) = \frac{1}{2i}\{((S^*T - T^*S)x|x) - (Tx|x)(x|Sx) + (Sx|x)(x|Tx)\}.$$

Remark 3.2. *We have*

$$\frac{1}{4}\|(S^*T - T^*S)x|x) - (Tx|x)(x|Sx) + (Sx|x)(x|Tx)\|^2 \leq \|Var_x(S)\| \cdot \|Var_x(T)\|.$$

This is an uncertainty relation for Hilbert C^*-module.

4 Parametrization of variance-covariance inequality from the point of q-statistics.

There is a recent topic of q-parametrization between Boson statistics and Fermion statistics. In this section we shall consider the q-parametrization of variance-covariance inequality. Then we remark a q-parametrization of Kantrovich inequality from the point of [FFNT]. First we shall review the fact of q-parametrization between Boson statistics and Fermion statistics [cf.2]. Fix a number $q \in [-1, 1]$. Take a separable complex Hilbert space $\mathcal{H}$. Let $\mathcal{F}^{\text{finite}}(\mathcal{H})$ be the linear space of the vector of the form $f_1 \otimes \cdots \otimes f_n \in \mathcal{H}^{\otimes n}$ $(n \in \mathbb{N}_0)$ and $\mathcal{H}^{\otimes 0} \simeq \mathbb{C}\Omega$ (Ω=the vacuum vector). We put the sesquilinear form $(\cdot, \cdot)_q$ on $\mathcal{F}^{\text{finite}}(\mathcal{H})$ defind by

$$< f_1 \otimes \cdots f_n, g_1 \otimes \cdots g_n >_q = \delta_{nm} \Sigma q^{i(\pi)} (f_1, g_{\pi(1)}) \cdots (f_h, g_{\pi(h)})$$

and put

$$i(\pi) = \sharp\{(i,j) | 1 \leq i < j \leq n, \pi(i) > \pi(j)\}.$$

Then we shall define the q-Fock space $\mathcal{F}_q(\mathcal{H})$ by $\overline{\mathcal{F}^{\text{finite}}(\mathcal{H})}^{(\cdot,\cdot)_q}$. using this q,

Definition 4.1. *For $A, B \in \mathcal{B}(\mathcal{H})$, we define the q-covariance for A, B in a state $x, x \in \mathcal{H}$, $\|x\| = 1$ by*

$$Cov_x^q(A, B) = (Ax \otimes x, Bx \otimes x)_q.$$

Under this setting ,

Proposition 4.2. *For $q \in [-1, 1]$, we have*

$$(Ax \otimes x, Bx \otimes x)_q = (Ax, x) + q(Ax, x) \cdot (x, Bx).$$

Corollary 4.3(Seo). *For $q = -1$,*

$$(Ax \otimes x, Bx \otimes x)_{-1} = (Ax \wedge x, Bx \wedge x) = Cov_x(A, B).$$

Corollary 4.4. *For $q = 1$,*

$$(Ax \otimes x, Bx \otimes x)_1 = (Ax \vee x, Bx \vee x) = (Ax, Bx) + (Ax, x)(x, Bx)$$

5 The bound of q-variances and q-covariances.

In this section we shall consider the bound of q-variances and q-covariances. Let $x \in \mathcal{H}$, $\|x\| = 1, q \in [-1, 1]$ and $0 < m \leq A \leq M$.

Then we have

$$Var_x^q(A) = (M + q(Ax, x))((Ax, x) - m) - ((M - A)(A - m)x, x) + m(1 + q)(Ax, x).$$

$$Var_x^q(A) \leq (M + q(Ax, x))((Ax, x) - m) + m(1 + q)(Ax, x).$$

We put

$$f(t) = (M + qt)(t - m) + m(1 + q)t.$$

Proposition 5.1. *For* $-\frac{M+m}{2M} < q \leq 1$,

$$Var_x^q(A) \leq M^2(1+q).$$

Proposition 5.2. *For* $-\frac{M+m}{2m} \leq -1 \leq q \leq -\frac{M+m}{2M}$,

$$Var_x^q(A) \leq -\frac{(M+m)^2}{4q} - Mm.$$

Using this ,we shall show the bound of q-covariances.

For $0 < m_1 \leq A \leq M_1$, $0 < m_2 \leq B \leq M_2$, we have the following.

Proposition 5.3. *For* $0 \leq q \leq 1$,

$$|Cov_x^q(A,B)| \leq M_1M_2(1+q).$$

Proposition 5.4. *Let* $-1 \leq q \leq 0$ *and* $-\frac{M_1+m_1}{2M_1} = -\frac{M_2+m_2}{2M_2}$.

(1) *For* $-\frac{M_1+m_1}{2M_1} < q$, $|Cov_x^q(A,B)| \leq M_1M_2(1+q)$.

(2) *For* $q \leq -\frac{M_1+m_1}{2M_1}$,

$$|Cov_x^q(A,B)| \leq \sqrt{\{-\frac{(M_1+m_1)^2}{4q} - M_1m_1\}\{-\frac{(M_2+m_2)^2}{4q} - M_2m_2\}}.$$

Proposition 5.5. *Let* $-1 \leq q \leq 0$ *and* $-\frac{M_1+m_1}{2M_1} \neq -\frac{M_2+m_2}{2M_2}$.

(1)*Let* $-\frac{M_2+m_2}{2M_2} < -\frac{M_1+m_1}{2M_1}$,

(α) *For* $-\frac{M_1+m_1}{2M_1} < q$,

$$|Cov_x^q(A,B)| \leq M_1M_2(1+q).$$

(β) *For* $-\frac{M_2+m_2}{2M_2} < q \leq -\frac{M_1+m_1}{2M_1}$,

$$|Cov_x^q(A,B)| \leq \sqrt{\{-\frac{(M_1+m_1)^2}{4q} - M_1m_1\}M_2^2(1+q)}.$$

(γ) *For* $q \leq -\frac{M_2+m_2}{2M_2}$,

$$|Cov_x^q(A,B)| \leq \sqrt{\{-\frac{(M_1+m_1)^2}{4q} - M_1m_1\}\{-\frac{(M_2+m_2)^2}{4q} - M_2m_2\}}.$$

(2) *For* $-\frac{M_1+m_1}{2M_1} < -\frac{M_2+m_2}{2M_2}$,

(α) *For* $-\frac{M_2+m_2}{2M_2} < q$,

$$|Cov_x^q(A,B)| \leq M_1M_2(1+q).$$

(β) *For* $-\frac{M_1+m_1}{2M_1} < q \leq -\frac{M_2+m_2}{2M_2}$,

$$|Cov_x^q(A,B)| \leq \sqrt{M_1^2(1+q)\{-\frac{(M_2+m_2)^2}{4q} - M_2m_2\}}.$$

(γ) *For* $q \leq -\frac{M_1+m_1}{2M_1}$,

$$|Cov_x^q(A,B)| \leq \sqrt{\{-\frac{(M_1+m_1)^2}{4q} - M_1m_1\}\{-\frac{(M_2+m_2)^2}{4q} - M_2m_2\}}.$$

Remark(q-Kantrovich inequality).

(1) *For* $q = 0$, $1 \leq \frac{M}{m}$.

(2) *For* $0 < q \leq 1$,

$$(Ax, x)(A^{-1}x, x) \leq \frac{M + Mq - m}{mq}.$$

(3) *For* $-1 \leq q < 0$, *and* $-\frac{m+M}{2M} < q$,

$$(Ax, x)(A^{-1}x, x) \leq \frac{M + Mq - m}{-mq}.$$

For $-1 \leq q < 0$ *and* $q \leq -\frac{m+M}{2M}$,

$$(Ax, x)(A^{-1}x, x) \leq \frac{1}{-q}\left(1 + \sqrt{\{-\frac{(M+m)^2}{4q} - Mm\}\{-\frac{(\frac{1}{m} + \frac{1}{M})^2}{4q} - \frac{1}{Mm}\}}\right).$$

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